

BLACK-BOX IDENTITY TESTING FOR LOW DEGREE UNMIXED $\Sigma\Pi\Sigma\Pi(k)$ CIRCUITS

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Abstract. A $\Sigma\Pi\Sigma\Pi(k)$ circuit $C = \sum_{i=1}^k F_i = \sum_{i=1}^k \prod_{j=1}^{d_i} f_{ij}$ is unmixed if for each $i \in [k]$, $F_i = f_{i1}(x_1) \cdots f_{in}(x_n)$, where each f_{ij} is a univariate polynomial given in the sparse representation. In this paper, we give a polynomial time black-box algorithm of identity testing for the low degree unmixed $\Sigma\Pi\Sigma\Pi(k)$ circuits. In order to obtain the black-box algorithm, we first show that a special class of low degree unmixed $\Sigma\Pi\Sigma\Pi(k)$ circuits of size s is $s^{O(k^2)}$ -sparse. Then we construct a hitting set \mathcal{H} in polynomial time for the low degree unmixed $\Sigma\Pi\Sigma\Pi(k)$ circuits from the sparsity result above. The constructed hitting set is polynomial size. Thus we can test whether the circuit or the polynomial C is identically zero by checking whether $C(a) = 0$ for each $a \in \mathcal{H}$. This is the first polynomial time black-box algorithm for the low degree unmixed $\Sigma\Pi\Sigma\Pi(k)$ circuits, which also partly answers a question of Saxena [16].

1. Introduction . A well known algebraic problem in algorithm design and complexity theory is the Polynomial Identity Testing (PIT) problem: given a multivariate polynomial $p(x_1, \dots, x_n)$ over a field \mathbb{F} , determine whether the polynomial is identically zero. In many situations, an arithmetic circuit C that computes the polynomial $p(x_1, \dots, x_n)$ is given as the input instead of the polynomial $p(x_1, \dots, x_n)$. Many other problems are related to PIT. For example, primality testing [1] or testing whether there is a perfect matching [12] reduces to test whether a particular polynomial is identically zero. Further, the proof of $IP = PSPACE$ [18] and the proof of the PCP theorem [4] in complexity theory rely on the identity testing.

There is a randomized polynomial time algorithm for PIT, which was given by Schwartz [17] and Zippel [20]. Later, several polynomial time randomized algorithms with fewer random bits were introduced [5, 11]. But it is open to derandomize those randomized polynomial time algorithms or design a deterministic polynomial time or subexponential time algorithms for PIT. Kabanets and Impagliazzo [6] proved that a polynomial time identity testing algorithm implies that either $NEXP \not\subseteq P/poly$ or Permanent is not computable by polynomial-size arithmetic circuits. For the historic reason, it is hard to show the arithmetic circuit lower bounds. Thus, researchers focus on PIT in some restricted circuit models. Identity testing for sparse polynomials were studied in [10]. There is a polynomial time algorithm if the sparsity of the circuit is polynomial bounded. Deterministic algorithms for some depth-3 circuits were known [8, 9]. Surveys [16, 19] have more information about the progress of the identity testing.

Agrawal and Vinay [2] showed that a complete derandomization of identity testing for depth-4 arithmetic circuits with multiplication gates of small fanin implies a nearly complete derandomization of general identity testing. As a result, it is important and meaningful to study the depth-4 arithmetic circuits. A polynomial $p(x_1, \dots, x_n)$ of degree $poly(n)$ is called a low degree polynomial. The arithmetic circuit C that computes a low degree polynomial is called a low degree circuit. Using the result of Raz and Shpilka [13], Saxena [15] gave a deterministic white-box algorithm for depth-4 diagonal $\Sigma\Pi\Sigma\Pi(k)$ circuits, which runs polynomial time for the low degree circuits. Saraf and Volkovich [14] recently presented a deterministic black-box polynomial time algorithm for the depth-4 multilinear $\Sigma\Pi\Sigma\Pi(k)$ circuits. Other results concerned with depth-4 circuits can be found in [19].

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The known algorithm for $\Sigma\Pi\Sigma\Pi(k)$ circuits whose multiplication gate have unmixed variables is non black-box. So it is interesting whether there are black-box algorithms for them. In fact, Sexena leaves this as an open problem in the survey [16]. In this paper, we resolve this problem for the low degree $\Sigma\Pi\Sigma\Pi(k)$ circuits.

1.1. Main Results. Similar to the result in [14], we first show that each multiplication gate (in the second level) of the pseudo-simple minimal low degree unmixed $\Sigma\Pi\Sigma\Pi(k)$ circuit is $s^{O(k^2)}$ sparse where s is the size of the circuit. Let $C = \sum_{i=1}^k F_i = \sum_{i=1}^k \prod_{j=1}^{d_i} f_{ij}$ be the $\Sigma\Pi\Sigma\Pi(k)$ circuit. Roughly speaking, C is pseudo-simple if there is no f_{ij} appears in all F_i . The formal definition will be given later. The circuit C is unmixed if for each $i \in [k]$, $F_i = f_{i1}(x_1) \cdots f_{in}(x_n)$, where each f_{ij} is a univariate polynomial that is given in the sparse representation. Based on this sparsity result, we obtain a polynomial time black-box algorithm for the low degree unmixed $\Sigma\Pi\Sigma\Pi(k)$ circuit.

1.2. Outlines. In section 2, we give required definitions, lemmas and theorems. The sparsity bound for the low degree unmixed $\Sigma\Pi\Sigma\Pi(k)$ circuit is given in section 3. We present the black-box identity testing algorithm in section 4.

2. Preliminaries.

2.1. Polynomials. The symbol $[n]$ denotes the set $\{1, \dots, n\}$. Let \mathbb{F} be the underlying field and let $\bar{\mathbb{F}}$ be its algebraic closure. We assume that \mathbb{F} contains sufficient number of elements. Let $\mathbb{F}[x_1, \dots, x_n]$ be a ring of polynomials with coefficients in \mathbb{F} . Given a nonzero polynomial $P \in \mathbb{F}[x_1, \dots, x_n]$, it can be written in exactly one way in the form

$$P = \sum_I \alpha_I x^I \quad (2.1)$$

where each coefficient $\alpha_I \neq 0$ and

$$x^I := x_1^{i_1} \cdots x_n^{i_n}$$

with $I = (i_1, \dots, i_n)$. The polynomial $P(x_1, \dots, x_n)$ depends on variable x_i if there are $c \in \bar{\mathbb{F}}^n$ and $b \in \bar{\mathbb{F}}$ such that $P(c_1, \dots, c_i, \dots, c_n) \neq P(c_1, \dots, b, \dots, c_n)$. Then define $\text{var}(P) := \{i : P \text{ depends on } x_i\}$. Let $P|_{x_A=a_A}$ be the polynomial with $x_i = a_i$ for every $i \in A \subseteq [n]$. Given a multi-index $I = (i_1, \dots, i_n)$, define $I_{A=0}$ to be the multi-index by setting $i_a = 0$ for every $a \in A$. We define the sparsity of the polynomial as follows.

DEFINITION 2.1. *The sparsity of the polynomial P is the number of (nonzero) monomials in P , which is represented by $\|P\|$. Given $A \subseteq [n]$, define $\|P\|_A = |\{I_{A=0} : \alpha_I \neq 0\}|$.*

If the polynomial contains a constant, assume that its multi-index I is $(0, \dots, 0)$. If x_i^0 is in a monomial, we can remove x_i^0 from the monomial. There is an example for the sparsity. Let $P = x_1^2 x_2^5 x_3 + x_1^3 x_2 x_3^6 + x_1 - x_1$, then $\|P\| = 2$ and $\|P\|_A = 2$ where $A = \{x_2, x_3\}$.

Given a subset $A = \{a_1, \dots, a_k\} \subseteq [n]$ and a multi-index $I = (i_1, \dots, i_n)$, define $I_A := (i_{a_1}, \dots, i_{a_k})$. We can eliminate the variables with zero index from each monomial in the polynomial. Then (2.1) can also be written as

$$P = \sum_{I_A} \alpha_{I_A} x^{I_A} \quad (2.2)$$

where $x^{I_A} = x_{a_1}^{i_{a_1}} \cdots x_{a_k}^{i_{a_k}}$.

LEMMA 2.2. *Let $\{F_i\}$ and $\{G_i\}$ be sets of polynomials in $\mathbb{F}[x_1, \dots, x_n]$ with $F_i, G_i \neq 0$. Then*

$$\|gcd(F_1 \cdot G_1, \dots, F_k \cdot G_k)\| \leq \|gcd(F_1, \dots, F_k)\| \cdot \|G_1\| \cdots \|G_k\|$$

The proof is in [14] (Observation 2.8).

The following lemma is a corollary of the Shearer's Lemma.

LEMMA 2.3. *Let $P \in \mathbb{F}[x_1, \dots, x_n]$ be a polynomial. Given k disjoint sets $A_1, \dots, A_k \subseteq [n]$ with $k \geq 2$, we have*

$$\|P\|^{k-1} \leq \prod_{j=1}^k \|P\|_{A_j}$$

The proof is in [14] (Corollary 2.6). Now we define an operator

DEFINITION 2.4. *Given $A \subseteq [n]$, $a \in \bar{\mathbb{F}}^n$ and $P, Q \in \mathbb{F}[x_1, \dots, x_n]$, define $D_{x_A=\alpha_A}(P, Q)$ as*

$$D_{x_A=\alpha_A}(P, Q) := P \cdot Q|_{x_A=\alpha_A} - P|_{x_A=\alpha_A} \cdot Q$$

2.2. Low Degree Circuits. A $\Sigma\Pi\Sigma\Pi(k)$ circuit C is a depth-4 circuit has four alternating layers of addition and multiplication gates and the number of input to the top addition gate is k . The $\Sigma\Pi\Sigma\Pi(k)$ circuit C with size s computes a polynomial in the form

$$C(x) = \sum_{i=1}^k F_i(x) = \sum_{i=1}^k \prod_{j=1}^{d_i} f_{ij}(x)$$

where each f_{ij} is a s -sparse polynomials (sparsity is at most s). The circuit C is called a low degree circuit, if it computes a polynomial $P \in \mathbb{F}[x_1, \dots, x_n]$ with degree at most $poly(n)$. Without loss of generality, we can replace $poly(n)$ with $O(n)$ or just n . For every $A \subseteq [k]$, define a subcircuit of C as $C_A := \sum_{i \in A} F_i$. The circuit C is minimal if $C_A \neq 0$ for each $\emptyset \subsetneq A \subsetneq [k]$. Given a polynomial $P \in \mathbb{F}[x_1, \dots, x_n]$, the circuit C is P -minimal if no proper subcircuit C_A has an indecomposable factor P .

Let C be the $\Sigma\Pi\Sigma\Pi(k)$ circuit whose multiplication gates have unmixed variables. Next, we define the pseudo greatest common divisors for the unmixed polynomials. Suppose that $F_1 = f_{11}(x_1) \cdots f_{1n}(x_n)$ and $F_2 = f_{21}(x_1) \cdots f_{2n}(x_n)$ where each $f_{ij}(x_j)$ is a monic polynomial, let $S_1 := \{f_{11}(x_1), \dots, f_{1n}(x_n)\}$ and $S_2 := \{f_{21}(x_1), \dots, f_{2n}(x_n)\}$. Then the pseudo greatest common divisor of F_1 and F_2 is

$$gcd(F_1, F_2)_{pseudo} = \prod_{f_i(x_i) \in S} f_i(x_i)$$

where $S = S_1 \cap S_2$. Similarly, let $S_i := \{f_{i1}(x_1), \dots, f_{in}(x_n)\}$ where $F_i = f_{i1}(x_1) \cdot f_{i2}(x_2) \cdots f_{in}(x_n)$ for $1 \leq i \leq d$ ($d > 2$). Define the pseudo greatest common divisor of F_1, \dots, F_d for $d > 2$ as

$$gcd(F_1, \dots, F_d)_{pseudo} = \prod_{f_i(x_i) \in S} f_i(x_i)$$

where $S = \bigcap_{i=1}^d S_i$. Generally, $\gcd(F_1, F_2)_{pseudo}$ is different from $\gcd(F_1, F_2)$. The unmixed circuit C is pseudo-simple if $\gcd(C)_{pseudo} = \gcd(F_1, \dots, F_k)_{pseudo} = 1$ where $C = \sum_{i=1}^k F_i$. The simplification of C is defined as $sim(C) := C / \gcd(C)_{pseudo}$. The polynomial $P \in \mathbb{F}[x_1, \dots, x_n]$ is said to be decomposable if it can be written as $P(X) = Q(X_A) \cdot G(X_{\bar{A}})$ where $\emptyset \subsetneq A \subsetneq [k]$. Otherwise, P is called indecomposable. The indecomposable factors of a polynomial $P \in \mathbb{F}[x_1, \dots, x_n]$ is $P = P_1(X_{I_1}) \cdot P_2(X_{I_2}) \cdots P_d(X_{I_d})$ such that all $\emptyset \subsetneq I_j \subsetneq [n]$ are disjoint sets of indices and the P_i -s are indecomposable for $1 \leq i \leq d$. Let $f|_{in}C$ denote that f is an indecomposable factor of C . The following lemma will be used to prove the sparsity bound.

LEMMA 2.5. *Let P be a non-constant univariate low-degree polynomial and let $Q \in \mathbb{F}[x_1, \dots, x_n]$ be a low-degree polynomial. Further, let $c \in \bar{\mathbb{F}}$ where $P(c) \neq 0$. Suppose that $var(P) = \{i\}$, then $D_{x_i=c}(P, Q) \equiv 0$ if and only if $P|_{in}Q$.*

Proof. Suppose that $D_{x_i=c}(P, Q) \equiv 0$. Since $P(c) \neq 0$, we have $P(c) = a$ where $a \in \bar{\mathbb{F}}$ is a nonzero element of the field. Thus from $D_{x_i=c}(P, Q) \equiv 0$, we have

$$P \cdot Q|_{x_i=c} = a \cdot Q$$

Since $Q|_{x_i=c}$ does not depend on x_i and P is a univariate polynomial, $P|_{in}Q$.

Now suppose that $P|_{in}Q$, we show $D_{x_i=c}(P, Q) \equiv 0$. Since $P|_{in}Q$, we have $Q = P \cdot H$ where P and H are variable disjoint factors of Q . Further, $P(c) = a$ for some nonzero $a \in \bar{\mathbb{F}}$. Thus

$$\begin{aligned} D_{x_i=c}(P, Q) &= P \cdot Q|_{x_i=c} - P|_{x_i=c} \cdot Q \\ &= P \cdot a \cdot H - a \cdot P \cdot H \\ &\equiv 0 \end{aligned}$$

□

The following lemma characterize the pseudo greatest common divisors.

LEMMA 2.6. *Let $P_1, \dots, P_k \in \mathbb{F}[x_1, \dots, x_n]$ be non-constant low degree unmixed polynomials. Let $P_i = \beta_i \cdot g_{i1}(x_1) \cdot g_{i2}(x_2) \cdots g_{in}(x_n)$ such that $\beta_i \in \mathbb{F}$ and $g_{i1}, g_{i2}, \dots, g_{in}$ are univariate monic polynomials for $2 \leq i \leq k$. Let $P_1 = \beta \cdot g_1(x_1) \cdot g_2(x_2) \cdots g_n(x_n)$ such that β is a field element and g_1, g_2, \dots, g_n are univariate monic polynomials. Then $\gcd(P_1, \dots, P_k)_{pseudo} \neq 1$ if and only if there exists a nonconstant factor $g_d(x_d)$ ($1 \leq d \leq n$) of P_1 such that $\prod_{1 \leq j \leq n} (g_d - g_{ij}) \equiv 0$ for each $2 \leq i \leq k$.*

Proof. Suppose $G := \gcd(P_1, \dots, P_k)_{pseudo} \neq 1$, then there exists a factor $g_d(x_d)$ of P_1 such that $g_d(x_d)|_{in}G$. Hence, $g_d(x_d)$ is a common factor of P_1, P_2, \dots, P_k . Then there exists $g_{id}(x_d)$ such that $g_d(x_d) - g_{id}(x_d) \equiv 0$ for each $2 \leq i \leq k$. Thus we have $\prod_{1 \leq j \leq n} (g_d - g_{ij}) \equiv 0$ for each $2 \leq i \leq k$.

Now suppose that there exists a factor $g_d(x_d)$ ($1 \leq d \leq n$) of P_1 such that $\prod_{1 \leq j \leq n} (g_d - g_{ij}) \equiv 0$ for each $2 \leq i \leq k$. Then there exists a j such that $g_d \equiv g_{ij}$ for each $2 \leq i \leq k$. Thus g_d is a common factor of P_1, P_2, \dots, P_k . As a result, g_d is a factor of $\gcd(P_1, \dots, P_k)_{pseudo}$. Since $g_d \neq 1$, we have $\gcd(P_1, \dots, P_k)_{pseudo} \neq 1$. □

2.3. Hitting Sets and Generators. A set $\mathcal{H} \subseteq \mathbb{F}^n$ is a hitting set for a circuit class \mathcal{M} , if given any non-zero circuit $P \in \mathcal{M}$, there exists $a \in \mathcal{H}$ such that $P(a) \neq 0$. A generator for the circuit class \mathcal{M} is a polynomial mapping $\mathcal{G} = (G_1, \dots, G_n) : \mathbb{F}^m \rightarrow \mathbb{F}^n$ such that for each nonzero n -variate polynomial $P \in \mathcal{M}$, we have $P(\mathcal{G}) \neq 0$. In the identity testing, generators and hitting sets play the same role. The following lemma is about the generator for the low degree polynomials.

LEMMA 2.7. *There is a generator $\mathcal{L}_s := (\mathcal{L}_{1,s}, \dots, \mathcal{L}_{n,s}) : \mathbb{F}^q \rightarrow \mathbb{F}^n$ for s -sparse low degree polynomials. The individual degrees of every $\mathcal{L}_{i,s}$ are bounded by $n-1$ and $q = O(\log_n s)$.*

The Lemma 2.7 follows from the following two facts.

FACT 1. *We have a hitting set \mathcal{H} with cardinality $\text{poly}(n, s, d)$ for each non-zero n -variate s -sparse polynomial with degree d over a field \mathbb{F} .*

The statement can be found in [10]. In particular, for each non-zero low degree n -variate s -sparse polynomial, there exists a hitting set with cardinality $\text{poly}(n, s)$.

FACT 2. *Let $|\mathbb{F}| > n$. Given a hitting set $\mathcal{H} \subseteq \mathbb{F}^n$ for a circuit class \mathcal{M} , there is a $\text{poly}(|\mathcal{H}|, n)$ time algorithm that produce a generator $\mathcal{G} : \mathbb{F}^q \rightarrow \mathbb{F}^n$ for \mathcal{M} . The individual degrees of each \mathcal{G}_i is bounded by $n - 1$ and $q = \lceil \log_n |\mathcal{H}| \rceil$.*

The proof can be found in [7]. Combining the Fact 1 and the Fact 2, we have the Lemma 2.7. Some additional facts are needed in the paper.

FACT 3. *Let $P = P_1 \cdot P_2 \cdots P_n$ be a product of nonzero polynomials where $P_i \in \mathcal{M}$ for each i . Let \mathcal{G} be a generator for \mathcal{M} , then $P(\mathcal{G}) \neq 0$.*

Another fact is given in [3].

FACT 4. *Let $P = P(x_1, \dots, x_n)$ be a polynomial in n variables over an arbitrary field \mathbb{F} . Suppose that the degree of P in x_i is bounded by d_i for $1 \leq i \leq n$. Let $S_i \subseteq \mathbb{F}$ be a set with at least $d_i + 1$ elements of \mathbb{F} . If $P = 0$ for all n -tuples $(a_1, \dots, a_n) \in \prod_{i=1}^n S_i$, then $P \equiv 0$.*

3. Upper Bound of the Sparsity. In this section, we give a sparsity upper bound for the pseudo-simple minimal low degree unmixed $\Sigma\Pi\Sigma\Pi(k)$ circuits that computes the zero polynomial. The proof is similar to that in [14] with some nontrivial modifications.

THEOREM 3.1. *Let C be a pseudo-simple minimal low degree unmixed $\Sigma\Pi\Sigma\Pi(k)$ circuit of size s , which can be written as $C(x) = \sum_{i=1}^k F_i(x)$. If C computes a zero polynomial, then $\|F_i\| \leq s^{5k^2}$ for each $i \in [k]$.*

Proof. We prove the statement by induction on k . Let $k = 2$ be the basis case. Since C is pseudo-simple and minimal, We have $C = c - c$ for some unit $c \in \mathbb{F}$. Thus the sparsity of F_1 and F_2 is one. Suppose that the statement is true for $2 \leq k \leq K - 1$, we show that the statement is true for $k = K$. The proof is based on three claims.

CLAIM 1. *Let $G := \gcd(F_1, \dots, F_t)_{\text{pseudo}}$ be the pseudo greatest common divisor of $\{F_1, \dots, F_t\}$, where $2 \leq t \leq k - 1$. The sparsity of G satisfies $\|G\| \leq s^{5(k-t+1)^2}$.*

Proof. Let $V := [n] - \text{var}(G)$, then we can write F_i for $1 \leq i \leq t$ as $F_i = G \cdot g_i$ where $\text{var}(g_i) \cap \text{var}(G) = \emptyset$. Further, we have $F_i = f_{i1}(x_1) \cdot f_{i2}(x_2) \cdots f_{in}(x_n)$ for $1 \leq i \leq k$. Let $M_{f_{1d}, F_i} := \prod_{1 \leq j \leq n} (f_{1d} - f_{ij})$ for all $1 \leq d \leq n$ and $2 \leq i \leq k$. Now define the polynomial

$$\Phi := \prod_{\emptyset \subsetneq A \subsetneq [k]} C_A \prod_{1d, i: M_{f_{1d}, F_i} \neq 0} M_{f_{1d}, F_i}$$

Since \mathbb{F} is sufficient large, there exists an element $a \in \bar{\mathbb{F}}^n$ such that $\Phi(a) \neq 0$. Hence $\Phi|_{x_V=a_V} \neq 0$. Let $F'_i = F_i|_{x_V=a_V}$ for each i and let $C' = C|_{x_V=a_V} = \sum_{i=1}^k F'_i$. We show that C' is pseudo-simple and minimal. Since $C_A|_{x_V=a_V} \neq 0$ for each nonempty proper subset A of $[k]$, C' is minimal. Since C is pseudo-simple, we have $\gcd(F_1, \dots, F_k)_{\text{pseudo}} = 1$. Hence for every non-constant factor f_{1d} of F_1 there exists $2 \leq i \leq k$ such that $M_{f_{1d}, F_i} \neq 0$ by the Lemma 2.6. Let $f'_{1d} = f_{1d}|_{x_V=a_V}$ for $1 \leq d \leq n$. Then for each f_{1d} of F_1 , there exists $2 \leq i \leq k$ such that $M_{f'_{1d}, F'_i} \neq 0$. Thus $\gcd(F'_1, \dots, F'_k)_{\text{pseudo}} = 1$. Define $P_1 := \sum_{i=1}^t F'_i$, then

$$P_1 = \left(\sum_{i=1}^t g_i|_{x_V=a_V} \right) \cdot G = c \cdot G$$

where $c \in \mathbb{F}$. Let $P_i := F'_{t+i-1}$ for $2 \leq i \leq k-t+1$. Now we define a new low degree unmixed $\Sigma\Pi\Sigma\Pi(k)$ circuit $C'' := \sum_{i=1}^{k-t+1} P_i$. At first, $C'' \equiv 0$, since $C' \equiv 0$. Because C' is minimal, $c \cdot G = C'_{[t]} \neq 0$ and $C''_A \neq 0$ for any nonempty subset $1 \notin A \subseteq [k-t+1]$. So C'' is minimal. Finally, C'' is pseudo-simple, since

$$\gcd(C'')_{pseudo} = \gcd(F'_1, \dots, F'_k)_{pseudo} = 1$$

Now we can apply the induction hypothesis, since $k-t+1 < k$. As a result, $\|G\| = \|P_1\| \leq s^{5(k-t+1)^2}$. \square

Let f be a univariate polynomial, the arithmetic circuit is called f -minimal if no proper subcircuit has an indecomposable factor f . Moreover, recall that $f|_{in} C$ means that f is an indecomposable factor of C .

CLAIM 2. *Let $C = \sum_{i=1}^k F_i$ be a low degree unmixed $\Sigma\Pi\Sigma\Pi(k)$ circuit computing the zero polynomial and let f be a univariate polynomial such that $f|_{in} F_k$ and $f \nmid_{in} F_1$. There exists a subset $B \subseteq [k]$ with $2 \leq |B| \leq k-1$ satisfying: the subcircuit C_B is f -minimal, $f|_{in} C_B$ and $1 \in B$.*

Proof. Since C computes the zero polynomial, we have

$$F_k = - \sum_{i=1}^{k-1} F_i$$

Further, since $f|_{in} F_k$, f is an indecomposable factor of $\sum_{i=1}^{k-1} F_i$. If $\sum_{i=1}^{k-1} F_i$ is not minimal, we can continuously partition $\sum_{i=1}^{k-1} F_i$ into f -minimal circuits such that $f|_{in} C_A$ for each f -minimal subcircuit C_A in the partition and F_i is contained in only one subcircuit of the partition for each $i \in [k-1]$. Suppose that C_B is the f -minimal subcircuit such that $1 \in B$ and $f|_{in} C_B$. Since $k \notin B$ and $f \nmid_{in} F_1$, we have $2 \leq |B| \leq k-1$. \square

CLAIM 3. *Let $C = \sum_{i=1}^t F_i$ be a pseudo-simple f -minimal low-degree unmixed $\Sigma\Pi\Sigma\Pi(k)$ circuit of size s where $2 \leq t \leq k-1$. Let f be a non-constant univariate low-degree polynomial. Suppose that $f|_{in} C$, then $\|F_1\|_{var(f)} \leq s^{5t^2}$.*

Proof. Suppose that $var(f) = \{d\}$ where $1 \leq d \leq n$ and there exists an element $a \in \mathbb{F}$ such that $f(a) \neq 0$. Let $C' := D_{x_d=a}(f, C) = \sum_{i=1}^t D_{x_d=a}(f, F_i)$. Since F_i is an unmixed polynomial, we have $F_i = P_i \cdot f_i$ where $d \notin var(P_i)$. If $f_{id}(x_d) \neq 1$ for F_i , then set $f_i := f_{id}(x_d)$. Otherwise let $f_i := 1$. Then by the Lemma 2.5, we have

$$C' = \sum_{i=1}^t P_i \cdot D_{x_d=a}(f, f_i) \equiv 0$$

Since C' is f -minimal, $C'_A = D_{x_d=a}(f, C'_A) \neq 0$ for each nonempty subset $A \subseteq [t]$. Thus C' is minimal. Further, since $D_{x_d=a}(f, f_i)$ is a constant, C' is a pseudo-simple low-degree unmixed $\Sigma\Pi\Sigma\Pi(k)$ circuit. Because $t \leq k-1$, we can apply the induction hypothesis. So $\|F_1\|_{var(f)} = \|P_1\| \leq s^{5t^2}$. \square

Now we can prove the induction step by contradiction. Without loss of generality, assume that $\|F_k\| > s^{5k^2}$. We show that $\|F_i\| \leq s^{5k^2-1}$ for $1 \leq i \leq k-1$. Since the circuit is symmetric, it is sufficient to prove that $\|F_1\| \leq s^{5k^2-1}$. Let $P_k := F_k / \gcd(F_1, F_k)_{pseudo}$. Without loss of generality, assume that $P_k = f_{k1}(x_1) \cdot f_{k2}(x_2) \cdots f_{km}(x_m)$. Then we have

$$\|P_k\| = \|f_{k1}(x_1) \cdot f_{k2}(x_2) \cdots f_{km}(x_m)\| \leq s^m$$

Further, we have

$$\|P_k\| \geq \frac{\|F_k\|}{\|\gcd(F_1, F_k)_{pseudo}\|} > s^{5k^2 - 5(k-1)^2}$$

by the Claim 1. Thus $m \geq 10k - 5$. Given any $f_{kj}(x_j)$ of P_k for $1 \leq j \leq m$, there exists a subset $A \subseteq [k]$ such that $1 \in A$, C_A is f_{kj} -minimal and $f_{kj}|_{in} C_A$ by the Claim 2. Assume that $A = \{1, 2, \dots, t\}$ where $2 \leq t \leq k-1$. Let $G = \gcd(F_1, \dots, F_t)_{pseudo}$, then $\|G\| \leq s^{5(k-t+1)^2}$ by the Claim 1. Since C_A is f_{kj} -minimal, $f_{kj} \nmid_{in} F_i$ for every $i \in [t]$. Thus $f_{kj} \nmid_{in} G$. Then $C'_A := \sum_{i=1}^t F_i/G$ is a pseudo-simple f_{kj} -minimal low-degree unmixed $\Sigma\Pi\Sigma\Pi(t)$ circuit. Since $f_{kj}|_{in} C'_A$, we have $\|F_1/G\|_{var(f_{kj})} \leq s^{5t^2}$ by the Claim 3. As a result, we have

$$\|F_1\|_{var(f_{kj})} \leq \|F_1/G\|_{var(f_{kj})} \cdot \|G\| \leq s^{5(k-1)^2 + k + 19}$$

The above inequality is valid for each $1 \leq j \leq 10k - 5$. Moreover, all f_{kj} -s share no variable. Then applying the Lemma 2.3, we have

$$\|F_1\| \leq \left(\prod_{j=1}^{10k-5} \|F_1\|_{var(f_{kj})} \right)^{\frac{1}{10k-6}} < s^{5k^2-1}$$

Since the circuit is symmetric, $\|F_i\| < s^{5k^2-1}$ holds for each $i \in [k-1]$. Hence,

$$\|F_k\| = \left\| \sum_{i=1}^{k-1} F_i \right\| \leq \sum_{i=1}^{k-1} \|F_i\| < s^{5k^2}$$

This leads to a contradiction. So $\|F_i\| \leq s^{5k^2}$ for each $i \in [k]$. \square

4. The Black-Box Algorithm. Similar to [14], we construct a generator for low degree unmixed $\Sigma\Pi\Sigma\Pi(k)$ circuits. The image of the generator is the hitting set for such circuits. Then a polynomial time black-box algorithm can be obtained from the generator. Fix a set $C = \{c_0, c_1, \dots, c_n\} \subseteq \mathbb{F}$ with $n+1$ distinct elements. Recall that \mathcal{L}_m is a generator for m -sparse low-degree polynomials. Let \vec{y}_i denote the vector with q entries for each i .

DEFINITION 4.1. For each $i \in [n]$ let $W_i(z) : \mathbb{F} \rightarrow \mathbb{F}$ be the degree n polynomial such that $W_i(c_j) = 1$ for $j \geq i$ and $W_i(c_j) = 0$ otherwise. For each $l \geq 1$, $m \geq 1$ and $i \in [n]$, define

$$\mathcal{S}_{l,m}^i(\vec{y}_1, \dots, \vec{y}_l, z_1, \dots, z_l) := \mathcal{S}_{l-1,m}^i(\vec{y}_1, \dots, \vec{y}_{l-1}, z_1, \dots, z_{l-1}) \cdot W_i(z_l) + \mathcal{L}_m^i(\vec{y}_l)(1 - W_i(z_l))$$

where $\mathcal{S}_{l,m}^i(\vec{y}_1, \dots, \vec{y}_l, z_1, \dots, z_l)$ is a function from $\mathbb{F}^{q \cdot l + l}$ to \mathbb{F} . Moreover, let $\mathcal{S}_{0,m}^i \equiv 0$ for any i . Then define $\mathcal{S}_{l,m} : \mathbb{F}^{q \cdot l + l} \rightarrow \mathbb{F}^n$ as

$$\mathcal{S}_{l,m} := (\mathcal{S}_{l,m}^1, \mathcal{S}_{l,m}^2, \dots, \mathcal{S}_{l,m}^n)$$

Now we have the following fact from the definition.

FACT 5. For each $0 \leq d \leq n$, we have

$$\mathcal{S}_{l,m} \mid_{z_l=c_d} = (\mathcal{S}_{l-1,m}^1, \dots, \mathcal{S}_{l-1,m}^d, \mathcal{L}_m^{d+1}, \dots, \mathcal{L}_m^n)$$

Then for every $\vec{a} \in \text{Im}(\mathcal{S}_{l-1,m})$ and $\vec{b} \in \text{Im}(\mathcal{L}_m)$, we have $(a_1, \dots, a_d, b_{d+1}, \dots, b_n) \in \text{Im}(\mathcal{S}_{l,m})$. In particular, $\text{Im}(\mathcal{S}_{l-1,m}) \cup \text{Im}(\mathcal{L}_m) \subseteq \text{Im}(\mathcal{S}_{l,m})$.

Next, we show that there is a generator for the low degree unmixed $\Sigma\Pi\Sigma\Pi(k)$ circuits

LEMMA 4.2. Suppose that $P \in \mathbb{F}[x_1, \dots, x_n]$ is a nonzero polynomial computed by a low degree unmixed $\Sigma\Pi\Sigma\Pi(k)$ circuit $C = \sum_{i=1}^k F_i = \sum_{i=1}^k \prod_{j=1}^{d_i} f_{ij}(x_j)$ with size $s \geq 2$ and $k \geq 1$. Then it holds that $P(\mathcal{S}_{k,m}) \neq 0$ for every $m \geq s^{5k^2+2}$.

Proof. At first, we need some claims that are needed in the proof.

CLAIM 4. Let $M \geq 1$ and let $C = \sum_{i=1}^k F_i$ be a low degree unmixed $\Sigma\Pi\Sigma\Pi(k)$ circuit of size s such that $\max_i \|F_i\| > M$. Let $a \in \mathbb{F}^n$ with $F_i(a) \neq 0$ for each $i \in [k]$. Then there is a $0 \leq t \leq n-1$ such that $M < \max_i \|F_i|_{x_{[t]}=a_{[t]}}\| \leq M \cdot s$.

Proof. Since $\max_i \|F_i\| > M$ and $\max_i \|F_i|_{x_{[n]}=a_{[n]}}\| \leq 1$, let t be the largest index such that $M < \max_i \|F_i|_{x_{[t]}=a_{[t]}}\|$. Thus we have $\max_i \|F_i|_{x_{[t+1]}=a_{[t+1]}}\| \leq M$. Since F_i is an unmixed polynomial and $F_i(a) \neq 0$, setting x_{t+1} to a_{t+1} can affect at most one factor $f_{ij}(x_j)$ (where $j = t+1$) for each F_i . As a result, it can reduce the sparsity by a factor at most $\|f_{ij}(x_j)\| \leq s$. Hence $\|F_i|_{x_{[t]}=a_{[t]}}\| \leq M \cdot s$. \square

CLAIM 5. Let $P \in \mathbb{F}[x_1, \dots, x_n]$ be a nonzero polynomial computed by a pseudo-simple minimal low-degree unmixed $\Sigma\Pi\Sigma\Pi(k)$ circuit $C = \sum_{i=1}^k F_i$ with size s and $k \geq 2$. In addition, let \mathcal{G}_{k-1} be a generator for $\Sigma\Pi\Sigma\Pi(k-1)$ circuits of size s and s -sparse polynomials. Then there is a $c \in \text{Im}(\mathcal{G}_{k-1})$ and $0 \leq t \leq n-1$ such that $P' := P|_{x_{[t]}=c_{[t]}}$ is a nonzero s^{5k^2+2} -sparse polynomials.

Proof. If $\max_i \|F_i\| \leq s^{5k^2}$, then $\|P\| \leq k \cdot s^{5k^2} \leq s^{5k^2+2}$. Otherwise, we have $\max_i \|F_i\| > s^{5k^2}$. Recall that $F_i = f_{i1}(x_1) \cdot f_{i2}(x_2) \cdots f_{in}(x_n)$ for $1 \leq i \leq k$. Let $M_{f_{1d}, F_i} := \prod_{1 \leq j \leq n} (f_{1d} - f_{ij})$ for all $1 \leq d \leq n$ and $2 \leq i \leq k$. Now define the polynomial

$$\Phi := \prod_{\emptyset \subsetneq A \subsetneq [k]} C_A \prod_{1d, i: M_{f_{1d}, F_i} \neq 0} M_{f_{1d}, F_i}$$

Since each multiplicand of Φ is either a s -sparse polynomial or a $\Sigma\Pi\Sigma\Pi(k-1)$ circuit, $\Phi(\mathcal{G}_{k-1}) \neq 0$. Thus there is a $c \in \text{Im}(\mathcal{G}_{k-1})$ such that $\Phi(c) \neq 0$. Since F_i appears in the multiplicands of Φ , $F_i(c) \neq 0$ for each $i \in [k]$. Then we have

$$s^{5k^2} < \max_i \|F_i|_{x_{[t]}=c_{[t]}}\| \leq s^{5k^2+1}$$

by the Claim 4. Let $C' := C|_{x_{[t]}=c_{[t]}} = \sum_{i=1}^k F_i|_{x_{[t]}=c_{[t]}}$. Similar to the argument in the proof of the Theorem 3.1, C' is pseudo-simple and minimal. Further, we have $C' \neq 0$ by the Theorem 3.1 and $\max_i \|F_i\| > s^{5k^2}$. Finally, $\|C'\| = \|P|_{x_{[t]}=c_{[t]}}\| \leq s^{5k^2} \cdot s \cdot k \leq s^{5k^2+2}$. \square

Now we prove the statement by induction on k . If $k = 1$, then P is a product of s -sparse polynomials. Thus by the Fact 3 and the the Fact 5, $P(\mathcal{S}_{1,m}) \neq 0$. Assume that the statement is true for $2 \leq k \leq K$. We can assume that C is pseudo-simple and minimal. If C is not minimal, there is a $\Sigma\Pi\Sigma\Pi(k-1)$ circuit C' computing P . Then the induction hypothesis can be applied. If C is not pseudo-simple, we have $P = G \cdot C'$ where $G = \gcd(C)_{\text{pseudo}}$ and $C' = \text{Sim}(C)$. Since G is a product of s -sparse polynomials, $G(\mathcal{S}_{k,m}) \neq 0$ by the reason that is identical to the base case. So without loss of generality, we can assume that C is pseudo-simple and minimal. By the induction hypothesis, $\mathcal{S}_{k-1,m}$ is a generator for $\Sigma\Pi\Sigma\Pi(k-1)$ circuits and it is also a generator for s -sparse polynomials. Then from the Claim 5, there is a $c \in \text{Im}(\mathcal{S}_{k-1,m})$

and $0 \leq t \leq n-1$ such that $P' = P|_{x_{[t]}=c_{[t]}}$ is a nonzero s^{5k^2+2} -sparse polynomial. Since \mathcal{L}_m is a generator for s^{5k^2+2} -sparse polynomials, there exists a $b \in \text{Im}(\mathcal{L}_m) \subseteq \text{Im}(\mathcal{S}_{k,m})$ such that $P'(b) \neq 0$. As a result, $P(a_1, \dots, a_t, b_{t+1}, \dots, b_n) \neq 0$. \square

Then a black-box algorithm can be obtained by constructing a hitting set for the low degree unmixed $\Sigma\Pi\Sigma\Pi(k)$ circuit.

Algorithm 1: Construct a Hitting Set

input : A low degree unmixed $\Sigma\Pi\Sigma\Pi(k)$ circuit C
output: A hitting set \mathcal{H}
1 begin
2 Construct a set $U \subseteq \mathbb{F}$ with size $n^3 + 1$;
3 $\mathcal{H} := \mathcal{S}_{k, s^{5k^2+2}}((U^q)^k \times U^k)$;
4 Return \mathcal{H} ;
5 end

THEOREM 4.3. *The Algorithm 1 is a polynomial time algorithm, which outputs a hitting set \mathcal{H} of size $n^{O(k)} \cdot s^{O(k^3)}$ for the low degree unmixed $\Sigma\Pi\Sigma\Pi(k)$ circuit with size s .*

Proof. Suppose that $P \in \mathbb{F}[x_1, \dots, x_n]$ is a nonzero low degree polynomial computed by the low degree unmixed $\Sigma\Pi\Sigma\Pi(k)$ circuit of size s . From the Lemma 4.2, $P(\mathcal{S}_{k, s^{5k^2+2}}) \neq 0$ that depends on $(q+1) \cdot k = k + k(5k^2 + 2) \log_n s$ variables. Since the individual degree is less than $n^3 + 1$ for $P(\mathcal{S}_{k, s^{5k^2+2}})$, there exists a $c \in \mathcal{H}$ such that $P(c) \neq 0$ by the fact 4. So \mathcal{H} is a hitting set of P . The size of \mathcal{H} is

$$|\mathcal{H}| \leq n^{O(k)} \cdot n^{O(k^3 \log_n s)} = n^{O(k)} \cdot s^{O(k^3)}$$

Since k is a constant and the generator for sparse polynomials can be constructed in polynomial time by the Fact 2, the generator $\mathcal{S}_{k, s^{5k^2+2}}$ can be constructed in polynomial time. Then it is obvious that the Algorithm 1 is a polynomial time algorithm. \square

5. Conclusions and an Open Problem. We give a polynomial time black-box algorithm of identity testing for the low degree unmixed $\Sigma\Pi\Sigma\Pi(k)$ circuits. An open problem related to our work is to design a polynomial time algorithm of identity testing for the general low degree $\Sigma\Pi\Sigma\Pi(k)$ circuits.

REFERENCES

- [1] M. AGRAWAL, N. KAYAL AND N. SAXENA, *Primes is in P*, The Annals of Mathematics, 160(2):781-793, 2004.
- [2] M. AGRAWAL AND V. VINAY, *Arithmetic circuits: A chasm at depth four*, In Proceedings of the 49th Annual IEEE Symposium on Foundations of Computer Science (FOCS), 67-75, 2008.
- [3] N. ALON, *Combinatorial Nullstellensatz*, Combinatorics, Probability and Computing, 8:7-29, 1999.
- [4] S. ARORA AND S. SAFRA, *Probabilistic checking of proofs: A new characterization of NP*, JACM, 45(1):70-122, 1998.
- [5] Z.Z. CHEN AND M.Y. KAO, *Reducing Randomness via irrational numbers*, In Proceedings of 29th Annual ACM Symposium on Theory of Computing (STOC), pp. 200-209, 1997.
- [6] V. KABANETS AND R. IMPAGLIAZZO, *Derandomizing Polynomial Identity Tests Means Proving Circuit Lower Bounds*, Computational Complexity, 13(1-2):1-46, 2004.

- [7] Z.S. KARNIN, P. MUKHOPADHYAY, A. SHPILKA, AND I. VOLKOVICH, *Deterministic identity testing of depth-4 multilinear circuits with bounded top fan-in*, In Proceedings of the 42nd Annual ACM Symposium on Theory of Computing (STOC), 649-658, 2010.
- [8] Z.S. KARNIN AND A. SHPILKA, *Black Box Polynomial Identity Testing of Generalized Depth-3 Arithmetic Circuits with Bounded Top Fan-In*, In Proceedings of the 23rd Annual Conference on Computational Complexity (CCC), 280-291, 2008.
- [9] N. KAYAL AND N. SAXENA, *Polynomial identity testing for depth 3 circuits*, Computational Complexity, 16(2):115-138, 2007.
- [10] A.R. KLIVANS AND D. SPIELMAN, *Randomness efficient identity testing of multivariate polynomials*, In Proceedings of the 33rd Annual ACM Symposium on Theory of Computing (STOC), 216-223, 2001.
- [11] D. LEWIN AND S. VADHAN, *Checking polynomial identities over any field: towards a derandomization?*, In Proceedings of the 30th Annual ACM Symposium on Theory of Computing (STOC), 438-447, 1998.
- [12] L. LOVÁSZ AND M.D. PLUMMER, *Matching Theory*, Elsevier, 1986.
- [13] R. RAZ AND A. SHPILKA, *Deterministic polynomial identity testing in non commutative models*, Computational Complexity, 14(1):1-19, 2005.
- [14] S. SARAF AND I. VOLKOVICH, *Black-box identity testing of depth-4 multilinear circuits*, ECCC, No.46, 2011.
- [15] N. SAXENA, *Diagonal circuit identity testing and lower bounds*, In Proceedings of 35th International Colloquium on Automata, Languages and Programming (ICALP), 60-71, 2008.
- [16] N. SAXENA, *Progress on Polynomial Identity Testing*, <http://www.math.uni-bonn.de/~saxena/papers/pit-survey09.pdf>
- [17] J. T. SCHWARTZ, *Fast Probabilistic Algorithms for Verification of Polynomial Identities*, JACM, 27(4):701-717, 1980.
- [18] A. SHEN, *IP = PSPACE: Simplified Proof*, JACM, 39(4):878-880, 1992.
- [19] A. SHPILKA AND A. YEHUDAYOFF, *Arithmetic circuits: A survey of recent results and open questions*, Foundations and Trends in Theoretical Computer Science, 5(3-4):207-388, 2010.
- [20] R. ZIPPEL, *Probabilistic algorithms for sparse polynomials*, Symbolic and Algebraic Computation, pp. 216-226, 1979.